THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Solution to Quiz 2

1. (a) (i) $AB > CD$ if there exists a point E on AB such that $A * E * B$ and $AE \cong CD$.

(ii) D is an interior point of an angle $\angle BAC$ if D, B are on the same side of the line l_{AC} and D, C are on the same side of the line l_{AB}.

(iii) $\Gamma = \{X : OX \cong OA\}$. B is said to be an interior point of Γ if $OB < OA$.

(b) By axiom **I3**, there exist three noncollinear points on the Hilbert plane, so we have at least one point D which is not O on the space.

By axiom **I1**, there exists a unique line l_{OD} containing O and D.

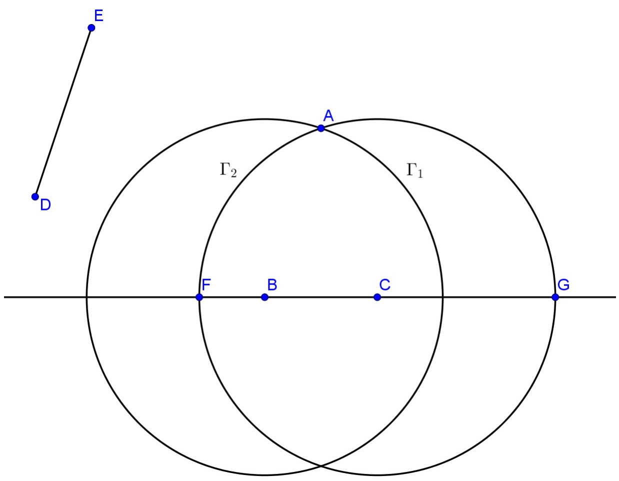
By axiom **C1**, there exists a unique point C on the ray r_{OD} oringinated from O such that $OC \cong AB$.

Then, the circle Γ with center O and radius OC is the required circle.

(c) By (b), we can construct two circles Γ_1 and Γ_2 such that their centers are B and C respectively and their radii are congruent to DE.

By axiom **I1**, there exists a unique line l containing C and B.

By axiom **C1**, the circle Γ_2 intersects the ray r_{CB} at a unique point F and intersects the ray opposite to r_{CB} at a unique point G.



Note that $CF \cong DE$ and $DE > BC$, therefore we have $F * B * C$ which means $CF > BF$. However, the radius of Γ_1 is congruent to DE as well as CF, therefore BF is less than the radius of Γ_1 and F is an interior point of Γ_1.

Also, by the construction of G, we know that G and B are on the opposite side of C. Therefore, we have $B * C * G$ which implies that $BG > CG$. However, the radius of Γ_1 is congruent to DE as well as CG, therefore BG is greater than the radius of Γ_1 and G is an interior point of Γ_1.

By axiom **E**, Γ_1 and Γ_2 will intersect. Suppose A is an intersection point, then $\triangle ABC$ is required triangle.

2. (a) Let $f(z) = \frac{1}{z}$. Then we have

$$\begin{aligned}
[f(z_1), f(z_2), f(z_3), f(z_4)] &= \left(\frac{f(z_4) - f(z_2)}{f(z_1) - f(z_2)}\right) / \left(\frac{f(z_4) - f(z_3)}{f(z_1) - f(z_3)}\right) \\
&= \left(\frac{\frac{1}{z_4} - \frac{1}{z_2}}{\frac{1}{z_1} - \frac{1}{z_2}}\right) / \left(\frac{\frac{1}{z_4} - \frac{1}{z_3}}{\frac{1}{z_1} - \frac{1}{z_3}}\right) \\
&= \left(\frac{\frac{z_2 - z_4}{z_2 z_4}}{\frac{z_2 - z_1}{z_1 z_2}}\right) / \left(\frac{\frac{z_3 - z_4}{z_3 z_4}}{\frac{z_3 - z_1}{z_1 z_3}}\right) \\
&= \left(\frac{z_2 - z_4}{z_2 - z_1}\right) / \left(\frac{z_3 - z_4}{z_3 - z_1}\right) \\
&= \left(\frac{z_4 - z_2}{z_1 - z_2}\right) / \left(\frac{z_4 - z_3}{z_1 - z_3}\right) \\
&= [z_1, z_2, z_3, z_4]
\end{aligned}$$

Therefore, $f(z) = \frac{1}{z}$ preserves four point ratios.

(b) we have

$$\begin{aligned}
[z_2, z_1, z_4, z_3] &= \left(\frac{z_3 - z_1}{z_2 - z_1}\right) / \left(\frac{z_3 - z_4}{z_2 - z_4}\right) \\
&= \left(\frac{z_2 - z_4}{z_2 - z_1}\right) / \left(\frac{z_3 - z_4}{z_3 - z_1}\right) \\
&= \left(\frac{z_4 - z_2}{z_1 - z_2}\right) / \left(\frac{z_4 - z_3}{z_1 - z_3}\right) \\
&= [z_1, z_2, z_3, z_4] \\
&= \lambda
\end{aligned}$$

Therefore, $[z_1, z_2, z_3, z_4]$ is real $\Leftrightarrow$ λ is real $\Leftrightarrow$ $[z_2, z_1, z_4, z_3]$ is real.

3. If $|z| = 1$, then

$$\begin{aligned}
|w|^2 &= w\overline{w} \\
&= \left(\lambda \frac{z - a}{\overline{a}z - 1}\right)\left(\overline{\lambda} \frac{\overline{z} - \overline{a}}{a\overline{z} - 1}\right) \\
&= \lambda\overline{\lambda} \frac{|z|^2 - a\overline{z} - \overline{a}z + |a|^2}{|a|^2|z|^2 - a\overline{z} - \overline{a}z + 1} \\
&= 1
\end{aligned}$$

The last equality follows from the fact that $|\lambda| = 1$ and the assumption that $|z| = 1$. Then, $|w|^2 = 1$ implies $|w| = 1$.